

What can Physics learn from Continuum Mechanics ?

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Dedicated to Prof. Ekkehart Kröner

Abstract

This paper is mostly a collection of ideas already published by various authors, some of them even a long time ago. Its intention is to bring the reader to know some rather unknown papers of different fields that merit interest and to show some relations between them the author claims to have observed. In the first section, some comments on old unresolved problems in theoretical physics are collected. In the following, I shall explain what relation exists between Feynman graphs and the teleparallel theory of Einstein and Cartan in the late 1920s, and the relation of both to the theories of the incompressible aether around 1840. Reviewing these developments, we will have a look at the continuum theory of dislocations developed by Kröner in the 1950s and some techniques of differential geometry and topology relevant for a modern description of defects in continuous media. I will then illustrate some basic concepts of nonlinear continuum mechanics and discuss applications to the above theories. By doing so, I hope to attract attention to the possible relevance of these facts for ‘fundamental’ physics.

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1 Old unresolved problems in theoretical physics

After the foundation of modern physics with its cornerstones quantum mechanics and general relativity up to 1930, theoretical physics has developed in a less revolutionary manner in the past decades. Richard Feynman mentioned in his Nobel lecture (1965) that he was driven by the hope to *calculate* the rest energy of an electron - which is an experimentally well-known quantity of 0.511 MeV. Not much has happened yet towards the solution of this problem. Another example where physicists seem to have surrendered regards the mass ratios $m_p/m_e = 1836.15\dots$, $m_n/m_e = 1838.68\dots$, $m_\mu/m_e = 206.768\dots$ of protons, neutrons, muons and others with respect to the electron. It seems like an evidence of incapacity for present day physics that the only attempts to calculate these numbers are some playing with powers of e 's and π 's without any physical background.¹ The same development has taken place with the fine structure constant $\alpha = 137.03597\dots$. Feynman commented:

'It is one of the *greatest* damn mysteries of physics. We know what kind of a dance to do experimentally to measure this number very accurately, but we don't know what kind of a dance to do on a computer to make this number come out - without putting it in secretly! ... all good theoretical physicists put this number on the wall and worry about it'.²

Today's physicists seem to prefer announcing some great unification now and then instead. Even more remote to a solution is the problem³ of the ratio of the electromagnetic and gravitational force, which is around 10^{40} . In this sense, physics hasn't moved closer towards a great unification since the speculations of Eddington (1929) and Dirac (1939). But should physicists disregard these problems forever?

Is there a reason why nature does not permit us to resolve these puzzles as a matter of principle, like the quadrature of the circle? If this should be the

¹How to play at least efficiently, one can read in Bailey and Ferguson (1989).

²Feynman 1985, chap. 4.

³This problem, which was already mentioned by Dirac (1939), was recently put in evidence by Weinberg (1999).

case, physicists havn't done their homework yet by proving their ' π ' to be transcendent.

Quantum electrodynamics had a great success after having calculated the the magnetic moment $1.00115965\mu_B$ of the electron and the Lamb shift of about 1040 MHz. However, in view of the above unsolved questions⁴, does this justify to build a general theory of physics on renormalization? Even Feynman himself was never convinced of the correctness of that theory:

'It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate.' (Feynman 1985, p. 128).

Dirac expressed himself more drastically: 'This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it because it is infinitely great and you do not want it.'⁵ However, since the times of QED a kind of monoculture of physical theories have been developed on its conceptual basis. Even Feynman commented self-ironically on the theories based on QED:

'so when some fool physicist gives a lecture at UCLA in 1983 and says: "This is the way it works, and look how wonderfully similar the theories are," it's not because nature is *really* similar; it's because the physicists have only been able to think of the same damn thing, over and over again.' (Feynman 1985, chap. 4, p.149).

Is it therefore not astonishing that general relativity remained 'off side' from the 'rest' of theoretical physics.⁶

In Ryder's (1985) book on quantum field theory we can read: 'the quantisation of the theory is beset by great problems...' ... 'in electrodynamics the field is

⁴As Feynman pointed out in his famous lectures (Feynman, R.B., and Sands 1963), the quantization of electrodynamics did not resolve the basic inconsistency of electrodynamics that predicts (with Coulomb's law and the energy density of the field) an infinite energy for the electron.

⁵cited by Kaku (1993), p. 12.

⁶Interestingly, experimenters face an embarrassing uncertainty ($1,5 \cdot 10^{-3}$) arising from discrepant measurements of the value of the gravitational constant G . Recently, Vargas and Torr (1999) suspected a theoretical reason for this.

an actor on the spacetime stage, whereas in gravity the actor becomes the spacetime stage itself.' Then the comment follows: ..'In view of this, the particle physicist is justified in ignoring gravity - and because of the difficulties mentioned above is happy to!' This kind of reasoning seems to be searching a key in the shine of a lamppost because one can see there, even if the key had been lost elsewhere in the dark.

In view of the unresolved problems physicists are rather justified to dig out and reconsider old ideas of excellent scientists - and because of the pleasure of reading their papers they should be happy to!

2 Lorentz symmetries in elastic solids

2.1 Frank's discovery

This section should guide the readers attention to a paper 'On the Equations of Motion of Crystal Dislocations' of Frank (1949). The abstract follows:

'It is shown that when a Burgers screw dislocation moves with velocity v it suffers a longitudinal contraction by the factor $\sqrt{1 - \frac{v^2}{c^2}}$, where c is the velocity of transverse sound. The total energy of the moving dislocation is given by the formula $E = E_0 / (1 - \frac{v^2}{c^2})^{\frac{1}{2}}$, where E_0 is the potential energy of the dislocation at rest. Taylor dislocations behave in a qualitatively similar manner, complicated by the fact that both longitudinal and transverse displacements and sound velocities are involved.'

A visualization of dislocations in crystals is given in fig. 2 in section 3.4. As has been pointed out by Frank, dislocations appear to have a particle-like behaviour. Their motion in a crystal close to the velocity of transverse sound is analogous to the motion of a particle close to the speed of light. Is this just a coincidence? At the first look, dislocations are a very special kind of defect. For deriving the above result, Frank considers the deformation field of a screw dislocation

$$u_x = 0, \quad u_y = 0 \quad u_z = (b/2\pi) \arctan \frac{y}{x}, \quad (1)$$

which is, since $\frac{\partial^2}{\partial x^2} u_z + \frac{\partial^2}{\partial y^2} u_z = 0$ and $\operatorname{div} \mathbf{u} = \frac{d}{dz} u_z = 0$, a statical solution of pure shear type of the Navier equation

$$-(\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \operatorname{curl} \operatorname{curl} \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad (2)$$

the displacement components $\mathbf{u} = (u_x, u_y, u_z)$ have to satisfy. For the following, it is sufficient to consider only the last two terms.

A dislocation propagating in x -direction with velocity v must be represented by a time-independent function of x' , y and z , where $x' = x - vt$. With this substitution, $\frac{\partial^2}{\partial t^2}$ becomes $v^2 \frac{\partial^2}{\partial x'^2}$, and the remaining terms of eqn. 2 read:

$$\mu \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + (\mu - v^2 \rho) \frac{\partial^2}{\partial x'^2} = 0 \quad (3)$$

With the further substitution

$$x'' = x' \sqrt{1 - \frac{v^2 \rho}{\mu}} = (x - vt) \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \quad (4)$$

where $c = \sqrt{\mu/\rho}$ is the velocity of transverse sound, the solution of a propagating dislocation has a form identical to the solution (1) of the dislocation at rest, apart from the substitution $x \rightarrow x'$, a 'Lorentz contraction'.

As Weertman and Weertman (1979), p. 8, commented: 'This distortion is analogous to the contraction and expansion of the electric field surrounding an electron.' Frank went ahead and showed that the elastic energy of the moving screw dislocation increases with the factor $(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$.

The question arises, whether this relativistic behaviour⁷ is a consequence of the special solution (1) or a more general effect. However, things would get more complicated only if the dilatational part $\operatorname{div} \mathbf{u}$

⁷**Radiation damping.** Interestingly, a phenomenon of radiation damping seems to occur in the dynamics of dislocations (Kosevic 1962; Kosevic 1979). That means, a part of the energy used in order to accelerate a dislocation is dissipated by the production of transversal sound waves. It should be noted that radiation damping in classical physics is everything but well understood. As e.g. (Dirac 1938), eq. 24, Landau and Lifshitz (1972), par. 75 or Feynman et al. (1963), chap. 28 point out, the Lorentz force $\vec{F}_L = e\vec{v} \times \vec{B}$ is just an approximation for small values of dv/dt , and a general formula for the radiation emitted by an accelerated electron does not exist. The quantization of electrodynamics didn't resolve this problem either.

of the displacement in eqn. 2 does *not* vanish. If it vanishes instead, the above transformation $x \rightarrow x''$ can obviously be applied to *every* solution of (2). The condition of the vanishing dilatation can be formally realized by letting go λ to infinity, which does physically mean that the medium is incompressible.⁸ The reader who is interested in details may look how other authors like Eshelby (1949), Weertman and Weertman (1979), Günther (1988), and in a somewhat redundant way, Günther (1996) have developed these analogies further, obtaining all features of special relativity including time dilatation etc. Thus, propagating solutions in an incompressible elastic continuum behave exactly as relativistic particles, if the speed of light is identified with the velocity of transverse sound.

2.2 MacCullagh's theory

Can these relativistic effects, apart from being a curiosity of elasticity theory, have a deeper meaning ? It does not seem so, because all attempts of describing fundamental physics with continuum mechanics in the 19th century have been falsified by the famous experiments by Michelson and Morley that seem to have disproved the concept of an aether. What's wrong here ? The point is, the physicists of the 19th century imagined particles as made of an external substance distinct from the 'aether', which for some reason can pass through it without (or with infinitely little) friction even with infinitely great velocities.

The aether theorists Young and Fresnel, Stokes, Navier, Cauchy, Lord Kelvin and Green never thought of particles as being topological defects creating a displacement field - which is not astonishing, since the first examples of such defects, dislocations in solids, were discovered in 1934 by Taylor, almost 30 years after aether theories had disappeared from the stage of theoretical physics. In view of the results of Frank (1949) and others, however, one must say that *the wrong concept was not describing spacetime as an elastic continuum but a wrong or missing picture of particles moving in it*. Therefore, theories of an incompressible aether like for example the one of MacCullagh (1839) do not contradict the the experiments of Michelson and Morley, if moving particles

⁸Of course, we are dealing with linear elasticity and have tacitly assumed small displacements. The nonlinear issues are discussed below.

are assumed to be propagating solutions.⁹

Let's have a look how MacCullagh¹⁰ identified the elecmagnetic quantities with those of an incompressible elastic aether:

According to this theory, one may identify the electric¹¹ field \vec{E} with the curl of the displacement field $\nabla \times \mathbf{u}$ and the magnetic field strength \vec{H} with its time derivative $\frac{d\mathbf{u}}{dt}$. Then, the Navier equation (2) reduces to

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \operatorname{curl} \operatorname{curl} \mathbf{u} \quad (5)$$

which is (μ is the shear modulus) equivalent to Maxwell's

$$\mu_0 \frac{\partial \vec{H}}{\partial t} = \frac{1}{\epsilon_0} \operatorname{curl} \vec{E}, \quad (6)$$

whereas $\operatorname{div} \vec{H} = 0$ follows directly from the incompressibility condition $\operatorname{div} \mathbf{u} = 0$ which implies $\operatorname{div} \frac{d\mathbf{u}}{dt} = 0$. By definition

$$\operatorname{div} \operatorname{curl} \mathbf{u} = 0 \quad \text{and} \quad \operatorname{curl} \frac{d}{dt} \mathbf{u} = \frac{d}{dt} \operatorname{curl} \mathbf{u} \quad (7)$$

holds, which correspond to Maxwell's second pair of equations *in vacuo*. Whittaker (1951), p. 143, comments:

It is evident from this equation (eqn. 5) that if $\operatorname{div} \mathbf{u}$ is initially zero it will be always zero; we shall suppose this to be the case, so that no longitudinal waves exist at any time in the medium. One of the greatest difficulties which beset elastic-solid theories is thus completely removed.¹²

Before I discuss in section 5.3 the topological issues how charged particles may enter in this model,

⁹As has been pointed out by Dirac (1951), aether theories do not contradict quantum mechanics either, as long as the absolute velocity of the aether material appears as a nonmeasurable quantity.

¹⁰This was the result presented in 1839 to the Royal Irish Academy by MacCullagh and published in 1848 in Trans. Roy. Irish Acad. xxi, p.17. The interested reader is referred to the excellent review on aether theories by Whittaker (1951), p. 142. ff; p. 280

¹¹To be precise, the electrostatic induction \vec{D} , which has to be divided by the dielectricity constant ϵ_0 to obtain \vec{E} .

¹²MacCullagh assumed furthermore the elastic energy to be a function of $\operatorname{curl} \mathbf{u}$. As we shall see later, this additional assumption is not necessary.

I would like to point out the connection with a paper of Einstein in 1930, where he states: '(46), (47) ... correspond to Maxwell's equations of empty space.' Einstein does not mention MacCullagh, although according to historians (Kostro 1992) he had given up his rejection to ether already around 1920. But it is better to tell the story from the beginning:

3 Einsteins teleparallel theory

3.1 The early papers 1928

There is an amazing contrast between the public admiration for Einstein for having developed general relativity and the importance that is given to his later work in differential geometry.¹³ More than once I happened to hear the statement that after 1920 Einstein had published just nonsense. Sympathetically, his work on teleparallel geometries has not been translated in English yet.¹⁴ Of course, there is a reason for the disregard of Einstein's work of the years after 1920: the conflict he had with quantum mechanics. His continuous objections, for example at the Solvay conference in 1927, could not unsettle the success of the new theory. On the contrary: people were realizing more and more that quantum mechanics was a good physical theory because it described the experiments, and got tired of the philosophical attacks launched by Einstein. In plain words: Einstein was a nuisance in the 1920s, and it is quite understandable that physicists were annoyed of the work that he proposed as an alternative to quantum mechanics and called unified field theory. Thus, why should we deal with his cumbersome tensor calculus developed in a series of papers¹⁵ and follow all his attempts that at the end were discarded by Einstein himself? A closer look at these papers, however, reveals that there is not only no contradiction to quantum mechanics on a conceptual level¹⁶, but there even arise some surprising facts from that geometry that remind us from the

¹³To avoid confusion, it should be mentioned that this theory distinguishes substantially from the so-called Einstein-Cartan-Sciama-Kibble (ECSK) theory. See Hehl, D.Kerlick, v.d.Heyde, and Nester (1976) for a review of several theories including torsion.

¹⁴A translation of some of his papers is available under www.lrz.de/~aunzicker/ae1930.html.

¹⁵Einstein (1928b); Einstein (1928a); Einstein (1930).

¹⁶Of course, the formalism is quite different from that of quantum mechanics, as that of GR is.

quantum behaviour of particles. I will discuss that in section 4. But let's listen to Einstein (1928b) now:

'Riemannian Geometry has led to a physical description of the gravitational field in the theory of general relativity, but it did not provide concepts that can be assigned to the electromagnetic field. Therefore, theoreticians aim to find natural generalizations or extensions of Riemannian geometry that are richer of concepts, hoping to get to a logical construction that unifies all physical field concepts under one single leading point.'

3.2 Torsion in Riemannian geometry

Einstein was convinced that the geometric description of physics does not stop at the rather special case of Riemannian geometry.¹⁷ In a later paper (Einstein 1930), he says:

'To take into account the facts (...) gravitation, we assume the existence of Riemannian metrics. But in nature we also have electromagnetic fields, which cannot be described by Riemannian metrics. The question arises: How can we add to our Riemannian spaces in a logically natural way an additional structure that provides all this with a uniform character?'

In the following Einstein refers to an idea that Cartan had pointed out to him already in 1922 - and Einstein did not understand at that time-, the 'Columbus connection':¹⁸ For Columbus, navigating straight meant going westwards. In terms of differential geometry: parallel transport of vectors means keeping a fixed angle to the lines of constant latitude, whereas usually the straight lines on a sphere are defined as the great circles¹⁹ (fig. 1).

¹⁷It should be mentioned that the notion of Riemannian geometry seems to have changed. Einstein intended a geometry in which the connection was determined by the metric only, with the absence of torsion (see also Schouten 1954,Bilby et al. 1955). Modern texts like Nakahara (1995) instead require just the existence of a Riemannian metric.

¹⁸Connection is the differential geometric entity that governs the law of parallel transport of vectors.

¹⁹Therefore, it is necessary to distinguish between *autoparallels*, on which vectors remain parallel, and *extremals*, that maximize the covered distance .

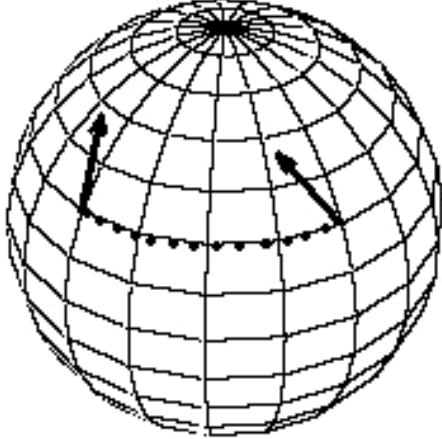


Figure 1: Visualization of the connection (vector transport rule) proposed by Cartan to Einstein. While transporting a vector (along the dotted line) the angle with the meridians is kept fixed. Thus, directions may be compared globally (Whenever we are speaking of ‘west’, ‘east’, ‘north’ and ‘south’, we are comparing directions globally!). If this is possible, a teleparallel connection can be given to the manifold and the curvature tensor vanishes.

Surprisingly, with this new connection the sphere has zero curvature but nonzero torsion.²⁰ If one looks at Fig.1, it becomes clear what Einstein said:

‘In every point there is a ... orthogonal n -bein²¹ . (...) The orientation of this n -beins is not important in a Riemannian manifold. We assume, that these (...) spaces are governed by still another direction law. We assume, (...) it makes sense to speak of a parallel orientation

²⁰One can imagine best the difference between curvature and torsion with differential forms. Both are 2-forms, that means quantities that have to be integrated over a 2-surface. If one transports a vector along a closed curve that bounds this surface, in the case of curvature it comes back *rotated*, and in the case of torsion *shifted*. Because this shift is done by a vector, torsion is called a *vector-valued* form, whereas curvature could be called a ‘rotation-valued’ form. Correctly speaking, it is a *Lie-algebra*-valued form. If the vector becomes just rotated (in the so-called metric-compatible case) the curvature form takes values in $so(3)$, the Lie algebra of orthogonal rotations in three-dimensional space. For an introduction to differential forms, see Flanders (1963) or Nakahara (1995).

²¹ n -‘leg’, from German ‘bein’, means n orthogonal unit vectors.

of all n -beins together (...).’

That was the idea that Einstein applied to space-time- describe the same physics with another differential geometric entity. Instead of nonzero curvature and vanishing torsion he proposed vanishing curvature and nonzero torsion - from the example fig. 1 it should be clear that this does not change the geometry of space. The advantage is that torsion in four dimensions has more components than curvature - that means one can pay the bill for describing gravity and hope that electromagnetism comes out of the additional components.

The problem was not that Einstein did not find tensor identities that were equivalent to Maxwell’s equations, he actually found too many of them - and nobody knows which identity is the right one that represents Maxwell’s equations - if there is any. For several reasons (see, e.g. Unzicker 1996, section 2.7), the proposed *field equations* (Einstein 1930, eqn. 29 and 30) must be wrong.²²

This does not imply, however, that the quantities he considered cannot have a reasonable meaning.

3.3 The electromagnetic field

We shall stop here as well for a moment and investigate what differential geometric quantities Einstein proposed for the electromagnetic field. In first approximation, he defines the electromagnetic field $a_{a\mu}$ in (Einstein 1930, eqn. 45) as

$$a_{a\mu} = \bar{h}_{a\mu} - \bar{h}_{\mu a}, \quad (8)$$

the antisymmetric part of the vielbeins $\bar{h}_{\mu a}$. The vielbeins h , as we shall see below, are nothing other than a generalization of the deformation gradient in continuum mechanics. In the case of a compatible deformation, the antisymmetric part defined in (8) is just the curl of the displacement vector \mathbf{u} - the same quantity²³ that had been proposed by MacCullagh !

Thus, this part of Einstein’s proposal was a kind of recycling MacCullagh’s old idea - I don’t know

²²In a letter to Salzer (1938, published in 1974), Einstein named as a reason for the failure of his teleparallel theory its representation of the electromagnetic field in first approximation, which does not transform as a tensor. We shall touch this problem in section 5.3.

²³Actually, it is not clear from Einstein’s paper whether he considered the $a_{a\mu}$ as the tensor of the electromagnetic field or its dual (\vec{E} and \vec{B} interchanged) - for Maxwell’s equations in empty space it makes no difference.

if he was aware of that and if he had liked it, if he were. It seems that at that time Einstein had given up denying the existence of an aether (Kostro 1992), but probably not because he was aware of that relation to MacCullagh's theory.

Einstens theory, however, is in a sense more general than MacCullagh's - Einstein's continuum cannot be described by a compatible deformation generated by a displacement field; this is a consequence of the nonvanishing torsion.

We will see that there remains a close relation between Einstein and MacCullagh as well. For this, a little excursion is needed to understand what torsion means.

3.4 Dislocations - a tool to understand torsion

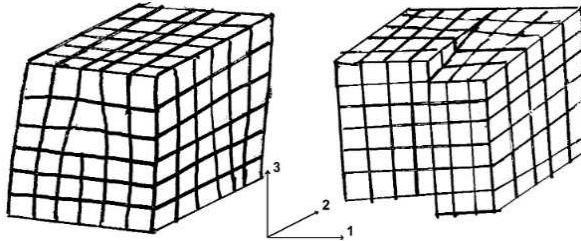


Figure 2: Examples of an edge (left) and a screw (right) dislocations in a crystal.

In 1952 Kondo revealed in an article of his wonderful review 'RAAG memoirs - the unifying study of basic problems in physics and engineering by means of geometry' the relation between dislocations and torsion. He discovered that torsion could be identified with a density of dislocations piercing through a surface element. The various components of torsion can be visualized in the example Fig. 2, an edge dislocation (left) and a screw dislocation (right) in a crystal. Suppose direction 1, 2, 3 point to the right, backwards and up as indicated. Then in the left picture, after surrounding a surface element in the 1-3-plane one gets shifted in direction 1, therefore this gives a contribution to the T_{13}^1 component of the torsion tensor. In the right picture, after surrounding a surface element in the 1-2 plane the shift is in direction 3, therefore this contributes to the T_{12}^3 component. Note that the singularity line of the dislocation in the left case goes in direction 2

and is perpendicular to the shift (Burgers vector), and in the right case parallel to the shift (both in direction 3). Torsion is just a continuous version of dislocation density, that means one lets the lattice spacing go to zero while maintaining the quantity shift per surface element.

Bilby, Bullough, and Smith (1955) have observed this equivalence independently and Kröner (1959; 1960) made a beautiful theory out of it.²⁴

Now, what can we learn from that ? On the one hand, that differential geometry with torsion is a good tool for describing dislocation behaviour in crystals.

On the other hand, we are able to give a physical interpretation to abstract geometries like those proposed by Einstein. In particular, there is no need to stick to the notion of a continuous torsion field. Spacetime could as well be endowed with a discrete torsion on a microscopic level that appears as dislocation density on the large scale. In this case it could be described by a compatible displacement field \mathbf{u} which has, however, singularities. Dislocations can be seen as singularities with Dirac-delta-valued torsion, but not all singularities in an elastic solid need to be dislocations. In section 5.3 I will discuss a topological defect that carries torsion without being a dislocation.

3.5 Cartan and topology

To be fair, one must say that Einstein did exclude that possibility and postulated a priori singularity-free solutions. Cartan, ²⁵ however, told him that postulating singularity-free solutions may create topological complications:

'As far as *singularity-free* solutions are concerned, it seems to me, the question is extremely difficult. (...) It is quite possible that the existence of singularity-free solutions imposes purely topological conditions on the continuum. (...) The

²⁴Kröner was fascinated by the similarities of this geometry and wrote: " We have seen that Riemannian geometry was too narrow to describe dislocations in crystals. Is there a reason why space-time has to be described by a connection that is less general than the general metric-compatible affine connection ?" (Kröner 1960, par. 18)

²⁵The interesting discussion between Einstein and Cartan is cited in the book by Debever (1979).

space in which the group exists, therefore depends *from the topological point of view*, on the constants Λ_{ij}^k (the torsion tensor), and every choice of the constants gives a space (or family of spaces) which is topologically defined. In short, *every singularity-free solution of system (1),²⁶ creates from the topological point of view the continuum in which it exists*. (letter to Einstein dated Jan 3rd, 1930)

Unfortunately, Einstein was not very interested²⁷ in the topological issues that arise in geometries with nonvanishing torsion:

'I cannot tell anything about the connectivity properties of space, but it seems unavoidable to demand singularity-free solutions...' (letter to Cartan dated Jan 30th, 1930)

Einstein's theory, however (or, in general, theories with torsion), allows an interpretation as geometry with a density of singularities on the microscopic level.

What do we gain with speculating about a discrete version of torsion and the interpretation as topological defects? I consider this interesting because it establishes a connection to a theory of physics that has been considered to be in blatant contradiction to Einstein's unified field theory - quantum mechanics. We shall see this in the following.

4 Topological defects

4.1 Quantum behaviour

Consider a pair of edge dislocations (as shown in fig. 2 a) in a two-dimensional view fig. 3. It is clear that to every such defect exists an antidefect (in this case, with the Burger's vector pointing in the opposite direction). If the two dislocations of opposite sign in the left and the right part of the picture start propagating towards each other, there will be

²⁶The equation $\Lambda_{\alpha\beta;\mu}^\gamma = 0$, whereby $\Lambda_{\alpha\beta}^\gamma$ is the torsion tensor.

²⁷For a discussion of these topics, in particular the Einstein-Cartan correspondence see also the papers by Vargas (1991; 1997; 1999).

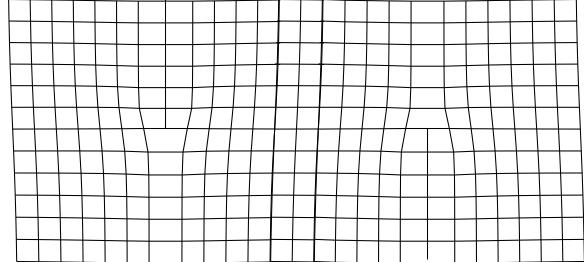


Figure 3: Two edge dislocations of opposite sign in a two-dimensional view. Their existence cannot be deduced by counting lattice points on the boundary as in the case of one single dislocation. If the two dislocations move towards each other, they will annihilate in the center.

an annihilation in the center. No topological irregularity of the lattice will be measurable, even if the elastic energy stored before will give rise to some lattice waves. If two dislocations as those in Fig. 3 move towards each other with a given velocity, it is even conceivable that the annihilation energy creates two other defects - not necessarily of the same structure. Thus, sticking to the particle picture, the encounter could be even seen as a scattering process, or it could appear as if the two dislocations pass through each other without interacting.

This doesn't seem extraordinary at all, but has some noteworthy consequences if we compare the motion of these defects with the motion of classical particles.

Fig. 4 a) shows the motion of a single dislocation propagating in x -direction from point P to Q . The slope in the $x-t$ diagram is a measure of its velocity. Analogously, fig. 4 can be interpreted as a Feynman Diagram for an electron propagating from P to Q (Feynman 1985, p. 99 and p. 125). The two signs of the dislocations correspond to the two signs of an electron and a positron; the latter one may be seen as an electron travelling backwards in time.

If one measures only the events P and Q , besides the 'direct path' Fig. 4 (a) the scenario (b) is possible as well: while propagating, the dislocation encounters its antidefect created by a spontaneous pair creation process and cancels out, whereas the other 'half' of the pair, identical to the original defect, continues propagating.

Of course, there may be many other scenarios

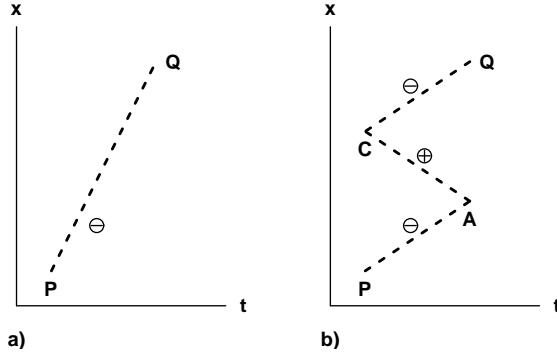


Figure 4: The propagation of dislocations is analogous to propagating electrons. When measuring the events in P and Q , there is no method to detect whether a defect propagating from P to Q goes a ‘direct’ path (a) or has a creation–annihilation process plugged in between (b). The ‘Feynman diagrams’ (a) and (b) are indistinguishable.

with the same experimental outcome corresponding to the various Feynman graphs (see Feynman 1985, p. 125).

Obviously, it doesn’t make sense to assign an ‘identity’ to this kind of ‘particles’. Once two defects have the same structure, they are identical. This kind of behaviour is well-known in quantum mechanics. Since particles are indistinguishable, they have to be described by Bose-Einstein or Fermi-Dirac statistics rather than by the classical Maxwell distribution.

Furthermore, it is clear that it makes no sense to speak about a ‘trajectory’ of the dislocation. This reminds us from the result of the double slit experiment that tells us that it makes no sense to say the electron passed through the one slit or the other.

Hence, dislocations behave not only relativistically, but also as quantum mechanical particles. Einstein, who introduced a geometry that describes dislocations may have been closer to the discovery of the puzzling quantum behaviour as he liked.

4.2 Homotopic classification of defects

In the above sections we have seen some examples of topological defects. For a precise definition of topological defects and for their classification, ho-

motopy theory is needed. It has been applied first by

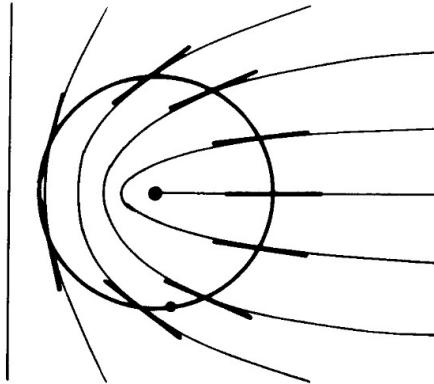


Figure 5: Topological defect in a nematic liquid crystal corresponding to the nontrivial element of the first homotopy group of the order parameter group RP^2 . The circle represents a path in RP^2 that cannot be contracted.

Toulouse and Kleman (1976) to defects in ordered media, good review articles are Rogula (1976), Mineev (1980), Michel (1980), Dzyaloshinskii (1980), Monastyrsky (1993) and Nakahara (1995), chap. 4.8. Ordered media have a so-called order parameter, in the example of fig. 5 the orientation of the bars in a nematic liquid crystal²⁸. Since this orientation is described by a *director*, the corresponding *order parameter space* is the projective plane RP^2 , which describes all possible orientations of these directors in space. In three dimensions, line defects (like the example fig. 5) can be detected by surrounding them by closed lines (homeomorphic to the circle S^1) and point defects by surfaces (homeomorphic to the sphere S^2). A topological defect occurs if the line or surface, mapped to the order parameter space, cannot be contracted to identity. Consequently, the first homotopy group consists of noncontractible loops in the order parameter space, and the second homotopy group to noncontractible closed surfaces. Nontrivial examples of the third homotopy group are more difficult to visualize, a quite comprehensive example (the Shankar monopole) is given by Nakahara (1995). The dislocations discussed above are line defects. The order parameter is the position of the 3-bein. Since after surrounding a defect this 3-bein is never rotated, the homotopy group consists of the

²⁸A fluid consisting of little rods.

possible shifts (by the discrete amount of the Burgers vector) in 3d-space, which is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Later I shall investigate the homotopy groups of $SO(3)$, the rotations in three-dimensional space.

5 Nonlinear continuum mechanics and spacetime analogies

5.1 Basic concepts

It has been known for a long time that the governing equations (2) of elasticity theory in the incompressible case lead to Maxwell's equations in empty space. As far as elasticity is considered, these equations are no more than a rather crude approximation - the classical linear theory that applies to small deformations. If one wants to push further the analogies between an elastic continuum and spacetime, there is no physical reason to assume the deformations to be small, in particular in the neighbourhood of topological defects which - in the view of section 2 - should take the part of elementary particles. Therefore, there is a quite natural option to generalize electrodynamics to a nonlinear theory²⁹ - just see how the real, nonlinear physics of elastic continua works.

Deformation gradient. Nonlinear elasticity goes back to the work of Cauchy in (1827). One assumes an undeformed, euclidean continuum with Cartesian coordinates $\mathbf{X} = (X, Y, Z)$ (the so-called 'reference configuration') and attaches in every point a displacement vector \mathbf{u} that points to the coordinates $\mathbf{x} = (x, y, z)$ of the deformed state ('configuration'): $\mathbf{u} = \mathbf{x} - \mathbf{X}$.³⁰

From \mathbf{u} one deduces the basic quantity in continuum mechanics that transforms the coordinates \mathbf{X} of the undeformed state to those \mathbf{x} of the deformed state: the so-called deformation gradient

$$\mathbf{F} := \frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

²⁹We should not forget the inconsistencies of classical linear electrodynamics mentioned above. Various attempts to modify electrodynamics are described in Feynman et al. (1963), chap. 28.5.

³⁰I use the common notion, e.g. in Truesdell and Toupin (1960), par. 15 ff., Truesdell and Noll (1965), or, in a more modern and didactical introduction, Beatty (1987).

or, in components,

$$\mathbf{F} := \begin{pmatrix} 1 + u_X & u_Y & u_Z \\ v_X & 1 + v_Y & v_Z \\ w_X & v_Y & 1 + w_Z \end{pmatrix}, \quad (9)$$

where (u, v, w) denote the components of \mathbf{u} and the subscripts differentiation.

It was Cauchy's merit to discover the importance of the symmetrical tensor

$$\mathbf{B} := \mathbf{F} \mathbf{F}^T, \quad (10)$$

which is now called *right* Cauchy-Green tensor.³¹

Polar decomposition. In tensor calculus it is a very common operation to split tensors in a symmetric and skew-symmetric part. In continuum mechanics, however, it would not make much sense to split the deformation gradient tensor in that way, because one could not assign the physical meaning of a deformation to the results any more. The above splitting would be a linear operation based on addition of matrices.

However, if a material undergoes a deformation \mathbf{F}_1 and then a successive deformation \mathbf{F}_2 , one has to *multiply* the matrices \mathbf{F}_1 and \mathbf{F}_2 to obtain the result \mathbf{F}_{12} - which is a noncommutative operation. Only for the small deformations (a case which is frequently assumed in linear elasticity) one can add $(\mathbf{F}_1 - \mathbf{E})$ and $(\mathbf{F}_2 - \mathbf{E})$ (\mathbf{E} is the unit matrix) and get an approximate result $\mathbf{F}_{12} - \mathbf{E}$.

Fortunately, mathematicians have developed the right way to split \mathbf{F} multiplicatively- it's the *polar decomposition*³² of

$$\mathbf{F} = \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{R},$$

where \mathbf{U} and \mathbf{V}^{33} are positive, symmetric matrices and \mathbf{R} is a rotation matrix, that means an element of the special orthogonal group $SO(3)$. Of course, a product in the above equation means matrix multiplication, and \mathbf{V} and \mathbf{U} are in general different because of their noncommutativity.

³¹ $\mathbf{C} := \mathbf{F}^T \mathbf{F}$ is called the left Cauchy-Green tensor.

³² \mathbf{F} has to be nonsingular.

³³This decomposition is unique. While one can find a proof of this theorem in many books, it is rarely told how to do it in practice. Surprisingly, there is no way to express the coefficients of \mathbf{R} and \mathbf{U} by the f_{ij} in general. Since the solution of this problem involves the zeros of the characteristic polynom of the matrix, the general formula of CARDANO makes the solution inhibitively complicated even for computer algebra systems.

Differential geometry. Even if there is very little overlap between the languages of books on elasticity and books on differential geometry³⁴, one should be aware of the similarity of some concepts. The deformation gradient \mathbf{F} corresponds to the basis 1-forms ϑ^i (cfr. Nakahara 1995, section 7.8). The difference is just that the ϑ^i maintain their meaning as quantities that transform coordinates, even if they cannot be deduced from a displacement field \mathbf{u} any more. Since $g_{\mu\nu} = \vartheta_\mu^i \vartheta_\nu^i$,³⁵ is equivalent to eqn. (10), the Cauchy tensor \mathbf{B} acquires the meaning of a metric. The so-called *compatibility* conditions which are necessary for the existence of a displacement field \mathbf{u} (see Vargas and Torr 1993 or Unzicker 1996), are expressed in differential geometry by the vanishing of both the curvature and the torsion tensor.

Since the Cauchy tensor \mathbf{B} and the stress tensor \mathbf{T} are both symmetric and coaxial, the meaning of a metric can be assigned to the latter tensor, too. An interesting comment on the duality of these tensors was given by Kröner (1986). Despite of the elegance of differential forms I do not know, however, how to express the above polar decomposition essential for nonlinear elasticity properly in that language.

5.2 Nonlinear extension of MacCullagh's proposal

Before going ahead, let's keep track of the quantities MacCullagh and Einstein had proposed for the electromagnetic field. The components of MacCullagh's *curl* \mathbf{u} or

$$\partial_i u^k - \partial_k u^i$$

are just the skew-symmetric part of the deformation gradient tensor \mathbf{F} (see eqn. 9). The same holds for the the antisymmetric part of the vielbeins $\bar{h}_{\mu a}$ (eqn. 8) Einstein proposed for the electromagnetic field.

Einstein was aware that this could be an approximation only, but he did not put in question the process of splitting the $h_{\mu a}$'s in a symmetric and skewsymmetric part. From MacCullagh's point of view, however, it would have been a rather natural option to pass from linear elasticity to the more general nonlinear equations.

Thus, if one wants to develop the analogy between spacetime and an elastic continuum as con-

sequent as possible, one should apply to the vielbeins $h_{\mu a}$ or the deformation gradient \mathbf{F} the polar decomposition theorem and identify the electric field with the rotational part \mathbf{R} , which takes values in $SO(3)$.³⁶

According to MacCullagh, the magnetic field corresponds to the velocity of the aether elements (Whittaker 1951, pp. 142, 280). Taking this into account, in the proposed nonlinear extension, the electromagnetic field takes values in $SO_0(3, 1)$, the connected component of the Lorentz group³⁷. Since this does not create further topological complications, I will sometimes restrict the discussion to $SO(3)$ in the following.

As mentioned above, MacCullagh's theory did not allow charges because of the vector analysis rule *div curl* = 0, applied to the displacement field \mathbf{u} . The nonlinear extension of MacCullagh's idea will help to overcome this difficulty by introducing a topological defect that acts as a 'source' of the rotations.

5.3 Topological defects as charges in MacCullagh's theory

Circular edge disclination. If we assume spacetime locally to be described by the deformation gradient \mathbf{F} , by the polar decomposition theorem $\mathbf{F} = \mathbf{R} \mathbf{U}$ follows that a unique field \mathbf{R} can be assigned to every point of spacetime.³⁸ In the language of the homotopic description of topological defects, one may regard \mathbf{R} , which takes values in $SO(3)$, as an order parameter field and ask about possible defects. Since the first homotopy group $\pi_1(SO(3))$ is \mathbb{Z}_2 , the group with two elements, mathematics predicts the existence of a line defect. The defect is then a closed singularity line, because defect lines cannot end inside the medium. In an elastic solid, this defect can be realized as follows:

One cuts the continuum along a (circular) surface, twists the two faces against each other by the amount of 2π and rejoins them again by gluing (Unzicker 1996).³⁹

³⁶For small fields, this is equivalent to MacCullagh's proposal *curl* \mathbf{u} .

³⁷The velocity of the aether material, according to special relativity, can be seen as a pseudorotation in the $x - t$ -plane.

³⁸If this is the case, a teleparallel connection can be defined, see also Truesdell and Noll (1965), par. 34.

³⁹The deformation gradient takes then infinite values at

³⁴An exception is Marsden and Hughes (1983).

³⁵see also Einstein 1928b, eq. (3), Einstein 1930, eq. (7).

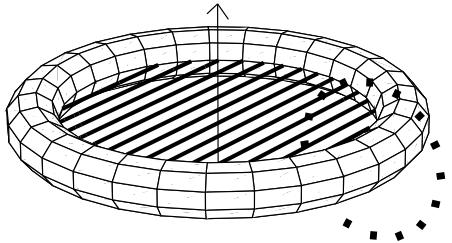


Figure 6: Scematic description how to produce the topological defect $1 \in \pi_1(SO(3)) = \mathbb{Z}_2$ in an elastic continuum. The solid torus is removed. Then the material is cut along the hatched surface. After twisting the cut faces by the amount of 2π , the material is rejoined. The dotted line represents a closed path in $SO(3)$ which is not contractible. To obtain a line defect, the solid torus can be shrunk to a singularity line. Note that after the cut, the same material elements are rejoined.

An alternative way to describe the same process is (see fig. 6): Imagine \mathbb{R}^3 filled with elastic material and remove a solid torus centered at the origin and with z as symmetry axis (fig. 6). Then the complement⁴⁰ is double connected due to the ‘bridge’ along the z -axis. One cuts now the material along the surface bounded by the inner circle of the torus in the $x - y$ -plane (hatched surface in fig. 6). Now the cut faces may be twisted, for example clockwise the face of the positive z -direction and counterclockwise the face of the negative one, and glued together again. If each of the twists amounts to π , the material elements meet their old neighbours again, so to speak, since the total twisting angle is 2π . To come back to the above description, now let the removed torus become infinitely thin.

A defect of this kind has been investigated first by Huang and Mura (1970), who called it edge the boundary of this circular surface.

⁴⁰For topologists, the complement is a solid torus as well, if \mathbb{R}^3 is compactified by adding the infinitely distant point. Two solid tori, merged in the described way, form the three-dimensional sphere S^3 .

disclination loop. Physically, it is more similar to a screw *dislocation*, even if closed loops of this kind do not exist (Unzicker 1996). Mura considered general twisting angles (Frank vector’s), whereas for topological reasons the twisting angle of 2π is the most interesting.⁴¹ To come back to homotopy theory, immagine now a closed path in the complement (the dotted line in fig. 6) that surrounds the removed torus, e.g. the singularity line. A twist of 2π in $SO(3)$ is equivalent to identity, thus the path is closed in $SO(3)$ as well. Since it is not contractible, the defect corresponds to the nontrivial element 1 of the first homotopy group \mathbb{Z}_2 . This topological description of the continuum defect has been given first by Rogula (1976).

Gaussian surface integral. We shall see now that the defect described above can be seen as an electrical charge in the nonlinear extension of MacCullagh’s theory. Except at the singularity line, a continuous field of the deformation gradient \mathbf{F} is given everywhere. Consequently, also the field \mathbf{R} obtained by polar decomposition is continuous, since in the region of the cut-and-glue surface the clockwise and the counterclockwise twist by the amount of π are the same element of $SO(3)$.

As the electrical field \vec{E} , one could regard an element of $SO(3)$ as having a ‘direction’ (ϑ, φ) (a point on the two-dimensional sphere S^2 which determines the rotation axis), and a ‘length’ r , the rotation angle⁴² ($0 \leq r \leq \pi$ and a direction in 3D-space $0 \leq \vartheta \leq \pi$; $0 \leq \varphi \leq 2\pi$)⁴³. However, even if this object has direction and length, it is not really a vector, since ‘vectors’ with opposite directions and length π are the *same* object.⁴⁴ This detail of replacing $\text{curl } \mathbf{u}$ by \vec{r} will avoid the consequence $\iint \text{curl } \mathbf{u} \, df = 0$ and make charges possible.

Consider a closed surface f that contains just the circular singularity line, like the surface shown in fig. 8. Then, in analogy to Gauss’s theorem, consider

⁴¹ Any twisting angle which is not a multiple of 2π would destroy the distant parallelism, i.e. the vielbeins could not be defined any more. Accordingly, disclination density is usually described by a nonvanishing curvature tensor.

⁴²this is sometimes called an ‘axial vector’.

⁴³This is called a representation of an abstract group like $SO(3)$. Another representation uses Euler angles (Love 1927).

⁴⁴It should further be noted that this ‘vector’ has *no* ‘components’, since due to the noncommutativity of $SO(3)$ the superposition principle does not hold.

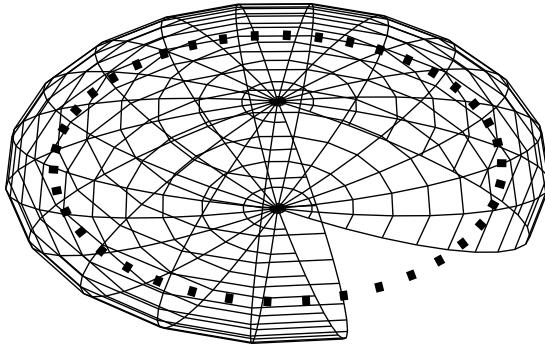


Figure 7: Example of a closed surface (for a better visualization, a sector is removed) surrounding the singularity line (dotted) generated by the shrinking of the torus in fig. 6. The Gaussian integral over this surface f is considered in the following.

the integral

$$\rho = \iint \vec{r} df, \quad (11)$$

where \vec{r} denotes the ‘vector’ in the above representation of $SO(3)$. If we shrink the surface f in fig. 7 to a minimal size in which it contains just the singularity line, \vec{r} points upwards for $z > 0$ and downwards for $z < 0$. If one integrates now *as if* \vec{r} would be a vector, the integral value is 2π times the area of the circle.

Note that the field \vec{r} *would be* discontinuous if it were a real vector field, but is continuous, if its values are correctly interpreted as elements of $SO(3)$. However, the nonzero value ρ of the integral appears to be a charge if \vec{r} is as usual identified with the electrical field \vec{E} . This charge is a consequence of the topological properties of the group of rotations in three-dimensional space. Hence, if MacCullagh’s idea is consequently extended by means of nonlinear continuum mechanics, electrically charged ‘particles’, i.e. topological defects become possible.

Quantization and screwsense. It is furthermore remarkable that any closed surface (like f in fig. 7) may contain only integer values of these defects, in accordance with the observed quantization of the electrical charge.

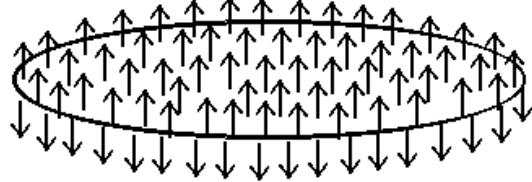


Figure 8: Visualization of the Gaussian integral $\rho = \iint \vec{r} df$ (The surface f in fig. 7 is shrunk to a minimal size that includes the singularity). If rotations in 3D-space are represented by a ‘vector’ \vec{r} , the rotations in the neighbourhood of the defect in fig. 6 can be represented as the vectors attached at the surface that contains the singularity line. ρ has nonzero value due to the fact that the discontinuous (at $z=0$) ‘vector’ field \vec{r} represents a continuous field of elements of $SO(3)$.

One may however raise the question how positive and negative charges that occur in nature are distinguished in this model, since the above defect corresponds to the element 1 of the first homotopy group \mathbb{Z}_2 .

Abstractly, two defects of \mathbb{Z}_2 would cancel out by the rule $1 + 1 = 0$. It should be noted, however, that this defect is not described completely by the first homotopy group. Coiled line defects like the above one do influence also higher homotopy groups (as π_2 is influenced by π_1 of the projective plane, see Nakahara (1995) and Mineev (1980)). For example, the above considerations on the integral eqn. 11 would hold also for a defect called Shankar’s monopole (Shankar 1977), representing the nontrivial element 1 of the *third* homotopy group $\pi_3(SO(3))$, which is \mathbb{Z} .

Apart from this it is important to note that the above defect can be realized in *two* different ways. Twisting clockwise for $z > 0$ and counterclockwise for $z < 0$ defines a screwsense, and if the twisting is done vice versa, the result is not the same defect but its mirror image. It is clear that a defect can be annihilated completely only by its mirror image, not by an identical defect. If two identical defects merge, the line singularity will disappear, but the result will be a nontrivial element of the third homotopy group \mathbb{Z} .⁴⁵

⁴⁵The second homotopy group is trivial.

5.4 Energy density in electromagnetism and continuum mechanics

Since $SO(3)$ is compact, the electrical field could take finite values⁴⁶ only. This becomes interesting if one assigns an energy density to the electromagnetic field. Then, integrating the energy density in the neighbourhood of an electron would not yield infinite energies as in classical electrodynamics.

If one respects however the analogy to nonlinear elasticity theory, the elastic energy⁴⁷ should not depend on the rotation \mathbf{R} . Rather it can be demonstrated that it depends on the eigenvalues of the Cauchy tensor only - this can be deduced by frame indifference and material symmetry considerations (Beatty 1987).

I have pointed out how MacCullagh's proposal of an incompressible elastic medium (Whittaker 1951, p.142) leads in linear approximation to Maxwell's equations. MacCullagh derived this by postulating the energy density function to depend on the rotation of the volume elements of the aether. This additional assumption, however, is not necessary since Maxwell's equations already follow from the incompressibility condition.

The question arises if the analogy between space-time and an elastic continuum can be developed as close as possible or if we are forced - contrarily to the theory of continuum mechanics - to assign an energy density that depends on the electric field, i.e. to the rotation of the volume elements.

But isn't the energy density $w = \frac{1}{2}\epsilon_0(E^2 + c^2B^2)$ of the electromagnetic field an experimental fact that compels us to choose the latter option? Since this formula, together with Coulomb's law, leads to the inconsistency of a infinite self-energy of the electron, it is interesting to see the comment given by Feynman on w :

'In practice, there are infinitely many possibilities for w and S (S is the Poynting vector) and up to now nobody has thought about an experimental method that allows to say which is the right one! People think that the most simple possibility is the right one, but we

⁴⁶Given an appropriate norm on $SO(3)$, like the absolute value r of the 'vector' mentioned above.

⁴⁷We do consider only the case in which an energy density can be defined, which is called the case of *hyperelasticity*. Truesdell and Noll (1965) advocate the more general case.

must admit that we don't know for sure which one describes the correct localization of the electromagnetic field energy in space.' (Feynman, R.B., and Sands 1963, chap. 27.4)

In view of this, one should note that in a deformed elastic solid it is obviously impossible that in a bounded region there is a nontrivial⁴⁸ field \mathbf{R} with a trivial Cauchy tensor \mathbf{B} ⁴⁹. Thus, the energy density one 'observes' for the electromagnetic field could be hidden in the deformation field, as elasticity theory says.

Strain-energy function. Let's review briefly how nonlinear continuum mechanics describes the localization of energy:

It is convenient to introduce the so-called *principal invariants* (I_1, I_2, I_3) of a tensor that are defined as follows:

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 \quad (\text{trace}) \quad (12)$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \quad (13)$$

$$I_3 = \lambda_1\lambda_2\lambda_3 \quad (\det) \quad (14)$$

(the λ_i are eigenvalues). Then, in general, the energy density W is a *function* of the principal invariants $W = W(I_1, I_2, I_3)$ of the Cauchy tensor, and the stress tensor⁵⁰ \mathbf{T} is given by⁵¹

$$\mathbf{T} = \beta_1\mathbf{B} + \beta_0\mathbf{E} + \beta_{-1}\mathbf{B}^{-1} \quad (15)$$

with the $\beta_i(I_1, I_2, I_3)$ being *functions* of the principal invariants of \mathbf{B} . For small deformations, i.e. for values of the I 's close to 1, the β 's have fixed values - that can be related to the known elastic *constants* in linear elasticity.⁵² Of course, this complications

⁴⁸'nontrivial' means also nonconstant here.

⁴⁹One would seek a theorem of matrix analysis that relates the global properties of these two fields; to my knowledge, it does not exist yet.

⁵⁰ \mathbf{T} is defined as traction per surface element. There are media with so-called microstructure where \mathbf{T} is not symmetric any more. The interested reader is referred to the papers of Mindlin (1964) and Toupin (1964).

⁵¹This is called the constitutive equation for isotropic materials. Note that there are no higher powers of \mathbf{B} . This is a consequence of the theorem of Cayley-Hamilton, see f.e. Beatty (1987).

⁵²As we see, nonlinear continuum mechanics has invented scalar-tensor-theories a long time ago. In Brans-Dicke theory (1963), the gravitational 'constant' G is a function of space-time, too.

need not to be realized in nature, but one should keep in mind the possibility that constants of nature are just the weak-field limit of field-dependent functions.

In conclusion, the nonlinear extension of the spacetime - elastic continuum analogy has interesting consequences for the localization of energy in space. This has to be developed and clarified further, but may become a possibility to overcome the inconsistencies classical electrodynamics still has to face.

Stopping these considerations at a necessarily incomplete stage, I'd like to mention the theoretical peculiarities that another element of MacCullagh's theory has as consequence - incompressibility.

5.5 Nonlinear theory of incompressibility

Notwithstanding its beauty, nonlinear elasticity has suffered from the fact that, due to its complexity, the calculation of concrete problems was inhibitively difficult. A big progress towards this direction was done in the late 1940s by Rivlin (1948), who - starting from rather practical problems given to him by the British rubber producer's association he worked for - developed many results of theoretical importance (Rivlin 1948).

The nonlinear condition of incompressibility is given by

$$I_3 = \lambda_1 \lambda_2 \lambda_3 = \det \mathbf{B} = (\det \mathbf{F})^2 = 1. \quad (16)$$

and *not*, as many texts on linear elasticity state, $\operatorname{div} \mathbf{u} = \mathbf{0}$. As a consequence, the elastic energy W depends on the principal invariants I_1 and I_2 of \mathbf{B} only.

Rivlin considered an energy density of the form

$$W(I_1, I_2) = C(I_1 - 3) + D(I_2 - 3) \quad (17)$$

which is the definition of a *Mooney-Rivlin* material⁵³ with the constants C and D . One should note that this kind of material is in a certain sense the most simple among the isotropic incompressible ones,⁵⁴

⁵³From incompressibility follows $I_3 = 1$.

⁵⁴Of course, one may further assume $C = D$, a case which is called Neo-Hookean.

but has still *two* elastic constants,⁵⁵ whereas in the corresponding linear case only *one* constant, f.e. the shear modulus μ describes the elastic properties of an isotropic incompressible material.⁵⁶

Rivlin's cube. Rivlin (1948; 1974) considered a cube of incompressible elastic material loaded uniformly by three identical pairs of equal and oppositely directed forces acting normally on its faces. The surprising result was (see, f.e. Beatty (1987), p. 1719 for a derivation) that besides the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 1$ there are six (!) others, from which three are stable and three inherently unstable. Hence, there might be the possibility of different stable solutions for the same topological defects, too.

Ericksen's problem. Determining the deformations arising from a given distribution of body forces and surface tractions for a material with arbitrary response functions β_i is possible for very few simplified cases. Therefore the attention of the theoreticians concentrated on deformations which arise from surface tractions alone. These deformations are called *controllable* and are also described by the equivalent condition of the vanishing divergence⁵⁷

⁵⁵It is not difficult to understand why incompressible nonlinear elasticity needs more than one 'shear modulus'. The reason is that squeezing and stretching are qualitatively different deformations. Consider a cube made of an incompressible elastic material under pressure on the $x - y$ faces. It will shorten in x -direction and, in order to preserve its volume, elongate in the x - and y - directions. In linear elasticity, the elastic energy depends just on the ratio of the length change in z -direction, which is the same for elongation and shortening. For large deformations, f.e. squeezing to the half of the height or doubling it by stretching, the nonvanishing components of the deformation gradient would be $f_{xx} = \lambda_1 = f_{yy} = \lambda_2 = \sqrt{2}$, $f_{zz} = \frac{1}{2}$ (squeezing) or $f_{xx} = \lambda_1 = f_{yy} = \lambda_2 = \frac{\sqrt{2}}{2}$, $f_{zz} = 2$ (stretching). The principal invariant I_1 (trace) of the Cauchy tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is therefore $2 + 2\frac{1}{4} = 4\frac{1}{4}$ in the first case and $\frac{1}{2} + \frac{1}{2} + 4 = 5$ in the latter (vice versa for I_2). Thus, nonlinear elasticity distinguishes between stretching and squeezing, or, in other words, elongation in one or two dimensions.

⁵⁶In linear elasticity, the incompressibility condition is $\sigma = \frac{1}{2}$ (Poisson's ratio), $k = \infty$ (compression modulus), $\lambda = \infty$ (Lame's constant) or $Y = 3\mu$ (Young's modulus). See Feynman, R.B., and Sands (1963), chap. 38; Love (1927), p. 103 for the definitions of the various constants. Only two of them (in the compressible case) are independent.

⁵⁷In practice, this is still not easy to calculate, since the derivations have to be taken with respect to the deformed, curvilinear coordinate system.

of the stress tensor $\text{div } \mathbf{T} = 0$.

The reason why dealing with controllable deformations is still a ‘dirty’ work is that the material-dependent response functions β_i still enter the calculations. However, in nonlinear elasticity exists an interesting class of solutions in which the β_i drop out of the final result, and these solutions are called ‘universal’.⁵⁸

Therefore, theoreticians were particularly interested in controllable deformations which are independent of the β_i (universal deformations), and Ericksen (1954) was the first to ask the question: ‘which deformations are possible in every isotropic, perfectly elastic body?’.

For *compressible* materials the answer (Ericksen 1955) is that only homogeneous deformations (that means with a constant deformation gradient \mathbf{F}) are possible. Surprisingly, for *incompressible* materials this is not the case. Ericksen gave examples of 4 different families of deformations and conjectured this classification to be complete. In the meantime, however, a fifth family has been detected, and it is still an open question if there are others or not (see Beatty (1987) and Saccomandi (2000) for a description of the families). This is just to give an example why the incompressible case is of theoretical interest.

5.6 Waves

Of course, the highly nonlinear condition eqn. 16 again complicates a lot the nice behaviour of eqn. (2) that led to linear electrodynamics. One important question is: May in the nonlinear case still waves exist that correspond to the electromagnetic waves we observe? Almost nothing is known about waves in the general case.

However, for the simpler case of eqn. (17) mentioned above some remarkable results hold:

”That is, *in a Mooney-Rivlin material subject to homogeneous strain, all disturbances parallel to a given transverse principal axis are propagated at a common speed and in unchanged form*. Perhaps this is a characterizing property

⁵⁸The most common and simple example is the simple shear of a rectangular block that in nonlinear elasticity cannot be produced by shear stresses only. Rather the shear stress is determined by the difference of the normal stresses only, see Truesdell and Noll (1965).

of the Mooney-Rivlin material; in any case, the possibility of waves of permanent form is certainly unexpected in a theory of finite deformation.” (Truesdell and Noll 1965, par. 95, p. 351 above).

Quite recently, Boulanger and Hayes (1994) and Boulanger (2000) have discovered that these soliton-like solutions which maintain their wave form while propagating exist also in the case of arbitrary, finite deformations.⁵⁹

That means that not only the classical behaviour of electromagnetic waves is recovered from nonlinear elasticity, but there are hints for a particle-like behaviour of propagating solutions coming from the nonlinear treatment.

6 Conclusions

The main purpose of this paper was to attack two popular preconceptions among today’s physicists.

The first one regards the compatibility of aether theories with the experimental facts of special relativity. It has been given evidence that not the concept of the aether as such is wrong, but the idea of particles consisting of external material passing through the aether. Rather the aether is a concept that yields special relativity in a quite natural way, provided that topological defects are seen as particles.

Independently from this, topological defects appear interesting, because they have been shown to behave as quantum mechanical particles under various aspects.

The second prejudice regards the compatibility of quantum mechanics with Einstein’s attempts of a unified field theory using teleparallelism. While there is no doubt that this theory presented in the stage around 1930 is wrong, I hope to have convinced the reader that it is worth to be studied as well. On the one hand, there is a very close relation –probably unknown to Einstein– to the theories of the incompressible aether, on the other hand Einstein’s theory anticipated the continuum theory of topological defects developed in the 1950s. Therefore, there is a clear possibility that quantum theory may emerge from the geometries Einstein con-

⁵⁹The authors investigated even the more general case of a previously deformed material.

sidered, even if his verbal attacks at that time still support today's common opinion of the incompatibility of his unified theory and quantum mechanics.

As a consequence, the author proposes to develop the far reaching analogies between spacetime and an elastic continuum in the most natural way - leaving the linear approximation and apply the nonlinear theory of finite deformations wherever possible.

The serious shortcut of MacCullagh's theory, the impossibility of electrical charges, can be overcome by applying the nonlinear theory. Other features of the general theory, like the localization of energy, appear promising or at least not contradictory to experimental facts.

Even if some topics necessarily have been discussed in a qualitative manner, the paper should contribute to help mathematics play a stronger role in theoretical physics.

In view of the open problems mentioned in section 1, the development of the theories that have come up in past decades seems to be rather exhausted.

One should therefore ask the question if physics can gain further insight from discussing events on the Planck scale or from the mathematics of differential geometry and homotopy groups.

A reconsideration of some old-fashioned physical theories with modern mathematics could be even more than a matter of historical interest that physicists and mathematicians of the 19th and the beginning 20th century merit.

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